ストークスの定理

定理 1.1 (ストークスの定理)**.** S をいくつかの閉じた曲線を境界とする向き付けられた曲面とする．この空間上でベクトル場 $\boldsymbol{E}$ は連続な偏導関数を持つとき，次が成り立つ:

$$\int_{\partial S} \boldsymbol{E} \cdot \boldsymbol{t} ds = \iint_S \mathrm{rot} \boldsymbol{E} \cdot \boldsymbol{n} dS \tag{1.1}$$

但し，$\boldsymbol{t}$ は ∂S の向きの単位接線ベクトル，$\boldsymbol{n}$ は S の向きの単位法線ベクトルとする．

証明. はじめに，曲面 S が,

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = x(u, v)\boldsymbol{i} + y(u, v)\boldsymbol{j} + z(u, v)\boldsymbol{k}, \tag{1.2}$$

とパラメータ表示されていて，曲面 S の法線ベクトル $\boldsymbol{n}$ は，S の向き付けと一致しているものとする．このとき曲面 S に対応する uv 平面の領域を D とすれば，曲面 S の境界 ∂S は領域 D の境界 ∂D が対応する．これより，

$$\int_{\partial S} \boldsymbol{E} \cdot \boldsymbol{t} ds = \int_{\partial S} \boldsymbol{E} \cdot d\boldsymbol{r} = \int_{\partial D} \left(\boldsymbol{E} \cdot \frac{\partial \boldsymbol{r}}{\partial u} du + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{r}}{\partial v} dv \right), \tag{1.3}$$

が成り立つ．従って，平面内のグリーンの定理を適用することにより，

$$\begin{aligned}
&\int_{\partial D} \left(\boldsymbol{E} \cdot \frac{\partial \boldsymbol{r}}{\partial u} du + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{r}}{\partial v} dv \right) \\
&= \iint_D \left[\frac{\partial}{\partial u} \left(\boldsymbol{E} \cdot \frac{\partial \boldsymbol{r}}{\partial v} \right) + \frac{\partial}{\partial v} \left(\boldsymbol{E} \cdot \frac{\partial \boldsymbol{r}}{\partial u} \right) \right] dudv \\
&= \iint_D \left[\frac{\partial \boldsymbol{E}}{\partial u} \cdot \frac{\partial \boldsymbol{r}}{\partial v} + \cancel{\boldsymbol{E} \cdot \frac{\partial^2 \boldsymbol{r}}{\partial u \partial v}} - \frac{\partial \boldsymbol{E}}{\partial v} \cdot \frac{\partial \boldsymbol{r}}{\partial u} - \cancel{\boldsymbol{E} \cdot \frac{\partial^2 \boldsymbol{r}}{\partial u \partial v}} \right] dudv \\
&= \iint_D \left[\frac{\partial \boldsymbol{E}}{\partial u} \cdot \frac{\partial \boldsymbol{r}}{\partial v} - \frac{\partial \boldsymbol{E}}{\partial v} \cdot \frac{\partial \boldsymbol{r}}{\partial u} \right] dudv \\
&= \iint_D \left[\left(\frac{\partial E_x}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial E_y}{\partial u} \frac{\partial y}{\partial v} + \frac{\partial E_z}{\partial u} \frac{\partial z}{\partial v} \right) - \left(\frac{\partial E_x}{\partial v} \frac{\partial x}{\partial u} + \frac{\partial E_y}{\partial v} \frac{\partial y}{\partial u} + \frac{\partial E_z}{\partial v} \frac{\partial z}{\partial u} \right) \right] dudv \\
&= \iint_D \Bigg[\left(\frac{\partial E_x}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial E_x}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial E_x}{\partial z} \frac{\partial z}{\partial u} \right) \frac{\partial x}{\partial v} \\
&\qquad + \left(\frac{\partial E_y}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial E_y}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial E_y}{\partial z} \frac{\partial z}{\partial u} \right) \frac{\partial y}{\partial v} \\
&\qquad + \left(\frac{\partial E_z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial E_z}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial E_z}{\partial z} \frac{\partial z}{\partial u} \right) \frac{\partial z}{\partial v} \\
&\qquad - \left(\frac{\partial E_x}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial E_x}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial E_x}{\partial z} \frac{\partial z}{\partial v} \right) \frac{\partial x}{\partial u} \\
&\qquad - \left(\frac{\partial E_y}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial E_y}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial E_y}{\partial z} \frac{\partial z}{\partial v} \right) \frac{\partial y}{\partial u} \\
&\qquad - \left(\frac{\partial E_z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial E_z}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial E_z}{\partial z} \frac{\partial z}{\partial v} \right) \frac{\partial z}{\partial u} \Bigg] dudv \\
&= \iint_D \Bigg[\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \left(\frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \frac{\partial y}{\partial v} \right) \\
&\qquad + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \left(\frac{\partial z}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial x}{\partial u} \frac{\partial z}{\partial v} \right) \\
&\qquad + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \right) \Bigg] dudv \\
&= \iint_D \left[\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \frac{\partial(y, z)}{\partial(u, v)} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \frac{\partial(z, x)}{\partial(u, v)} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \frac{\partial(x, y)}{\partial(u, v)} \right] dudv \\
&= \iint_S \left[\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) dydz + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) dzdx + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) dxdy \right] \\
&= \iint_S \mathrm{rot} \boldsymbol{E} \cdot \boldsymbol{n} dS,
\end{aligned}$$

より，示せた．一般の曲面については，各領域が上の証明で用いたのと同じようになるように分割すればよい． □