微分演算子の球座標表示を求める

まず最初に，3 次元デカルト座標から球座標への変換公式は次の通りであった:

$$x = r\sin\theta\cos\varphi, \tag{1}$$
$$y = r\sin\theta\sin\varphi, \tag{2}$$
$$z = r\cos\theta, \tag{3}$$

これより，簡単に，

$$r^2 = x^2 + y^2 + z^2, \tag{4}$$
$$\tan\theta = \frac{\sqrt{x^2+y^2}}{z}, \tag{5}$$
$$\tan\varphi = \frac{y}{x}, \tag{6}$$

が得られる．

まず，(4) を x で偏微分して，

$$2r\frac{\partial r}{\partial x} = 2x, \tag{7}$$

より，

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \sin\theta\cos\varphi, \tag{8}$$

を得る．y, z の偏微分については，対称性より途中まで同じだから，

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta\sin\varphi, \tag{9}$$
$$\frac{\partial r}{\partial z} = \frac{z}{r} = \cos\theta, \tag{10}$$

となる．次に，(5) を x で偏微分して，

$$\frac{1}{\cos^2\theta}\frac{\partial\theta}{\partial x} = \frac{x}{z\sqrt{x^2+y^2}}, \tag{11}$$

より，

$$\frac{\partial\theta}{\partial x} = \cos^2\theta\times\frac{x}{z\sqrt{x^2+y^2}} = \cos^2\theta\times\frac{r\sin\theta\cos\varphi}{r\cos\theta\cdot r\sin\theta} = \frac{\cos\theta\cos\varphi}{r} \tag{12}$$

y についても同様にして，

$$\frac{\partial\theta}{\partial y} = \cos^2\theta\times\frac{y}{z\sqrt{x^2+y^2}} = \cos^2\theta\times\frac{r\sin\theta\sin\varphi}{r\cos\theta\cdot r\sin\theta} = \frac{\cos\theta\sin\varphi}{r} \tag{13}$$

一方，z については，

$$\frac{1}{\cos^2\theta}\frac{\partial\theta}{\partial z} = -\frac{\sqrt{x^2+y^2}}{z^2}, \tag{14}$$

だから，

$$\frac{\partial\theta}{\partial z} = -\cos^2\theta\times\frac{\sqrt{x^2+y^2}}{z^2} = -\cos^2\theta\times\frac{r\sin\theta}{r^2\cos^2\theta} = -\frac{\sin\theta}{r}, \tag{15}$$

を得る．

次に，(6) を x で偏微分して，

$$\frac{1}{\cos^2\varphi}\frac{\partial\varphi}{\partial x} = -\frac{y}{x^2}, \tag{16}$$

が得られるから，

$$\frac{\partial\varphi}{\partial x} = -\cos^2\varphi\times\frac{y}{x^2} = -\cos^2\varphi\times\frac{r\sin\theta\sin\varphi}{r^2\sin^2\theta\cos^2\varphi} = -\frac{\sin\varphi}{r\sin\theta}, \tag{17}$$

となる．一方，y については，

$$\frac{\partial\varphi}{\partial y} = \cos^2\varphi \times \frac{1}{x} = \frac{\cos\varphi}{r\sin\theta}, \tag{18}$$

となる．また明らかに，

$$\frac{\partial\varphi}{\partial z} = 0, \tag{19}$$

である．以上より，任意の関数 f について，

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial\theta}\frac{\partial\theta}{\partial x} + \frac{\partial f}{\partial\varphi}\frac{\partial\varphi}{\partial x} = \sin\theta\cos\varphi\frac{\partial f}{\partial r} + \frac{\cos\theta\cos\varphi}{r}\frac{\partial f}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial f}{\partial\varphi}, \tag{20}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial\theta}\frac{\partial\theta}{\partial y} + \frac{\partial f}{\partial\varphi}\frac{\partial\varphi}{\partial y} = \sin\theta\sin\varphi\frac{\partial f}{\partial r} + \frac{\cos\theta\sin\varphi}{r}\frac{\partial f}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial f}{\partial\varphi}, \tag{21}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial f}{\partial\theta}\frac{\partial\theta}{\partial z} + \frac{\partial f}{\partial\varphi}\frac{\partial\varphi}{\partial z} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}, \tag{22}$$

が成り立つから，微分演算子は f を取り除いた，

$$\frac{\partial}{\partial x} = \sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}, \tag{23}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}, \tag{24}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \tag{25}$$

となることになる．

ラプラシアン $\Delta \overset{\text{def}}{=} \nabla^2$ の球座標表示

前節の結果より，

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \left(\frac{\partial}{\partial x}\right)^2 \\
&= \left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right) \\
&= \left(\sin\theta\cos\varphi\frac{\partial}{\partial r}\right)\cdot\left(\sin\theta\cos\varphi\frac{\partial}{\partial r}\right) + \left(\frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta}\right)\cdot\left(\frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta}\right) + \left(-\frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\left(-\frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right) \\
&+ \left(\sin\theta\cos\varphi\frac{\partial}{\partial r}\right)\cdot\left(\frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta}\right) + \left(\frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta}\right)\cdot\left(\sin\theta\cos\varphi\frac{\partial}{\partial r}\right) \\
&+ \left(\frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta}\right)\cdot\left(-\frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right) + \left(-\frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\left(\frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta}\right) \\
&+ \left(-\frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\left(\sin\theta\cos\varphi\frac{\partial}{\partial r}\right) + \left(\sin\theta\cos\varphi\frac{\partial}{\partial r}\right)\cdot\left(-\frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right) \\
&= \sin^2\theta\cos^2\varphi\frac{\partial^2}{\partial r^2} + \frac{\cos^2\theta\cos^2\varphi}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\sin^2\varphi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \\
&+ \frac{2\sin\theta\cos\theta\cos^2\varphi}{r}\frac{\partial^2}{\partial r\partial\theta} - \frac{2\cos\theta\sin\varphi\cos\varphi}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi} - \frac{2\sin\varphi\cos\varphi}{r}\frac{\partial^2}{\partial\varphi\partial r} \\
&+ \frac{\cos^2\theta\cos^2\varphi}{r}\frac{\partial}{\partial r} + \frac{\sin^2\varphi}{r}\frac{\partial}{\partial r} + \frac{\cos\theta\sin^2\varphi}{r^2\sin\theta}\frac{\partial}{\partial\theta} - \frac{\sin\theta\cos\theta\cos^2\varphi}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos^2\theta\sin\varphi\cos\varphi}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi} + \frac{\sin\varphi\cos\varphi}{r^2}\frac{\partial}{\partial\varphi}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial y^2} &= \left(\frac{\partial}{\partial y}\right)^2 \\
&= \left(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\left(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right) \\
&= \left(\sin\theta\sin\varphi\frac{\partial}{\partial r}\right)\cdot\left(\sin\theta\sin\varphi\frac{\partial}{\partial r}\right) + \left(\frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta}\right)\cdot\left(\frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta}\right) + \left(\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\left(\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right) \\
&+ \left(\sin\theta\sin\varphi\frac{\partial}{\partial r}\right)\cdot\left(\frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta}\right) + \left(\frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta}\right)\cdot\left(\sin\theta\sin\varphi\frac{\partial}{\partial r}\right) \\
&+ \left(\frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta}\right)\cdot\left(\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right) + \left(\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\left(\frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta}\right) \\
&+ \left(\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\left(\sin\theta\sin\varphi\frac{\partial}{\partial r}\right) + \left(\sin\theta\sin\varphi\frac{\partial}{\partial r}\right)\cdot\left(\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right) \\
&= \sin^2\theta\sin^2\varphi\frac{\partial^2}{\partial r^2} + \frac{\cos^2\theta\sin^2\varphi}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cos^2\varphi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \\
&+ \frac{2\sin\theta\cos\theta\sin^2\varphi}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{2\cos\theta\sin\varphi\cos\varphi}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi} + \frac{2\sin\varphi\cos\varphi}{r}\frac{\partial^2}{\partial\varphi\partial r} \\
&+ \frac{\cos^2\theta\sin^2\varphi}{r}\frac{\partial}{\partial r} + \frac{\cos^2\varphi}{r}\frac{\partial}{\partial r} - \frac{\sin\theta\cos\theta\sin^2\varphi}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos\theta\cos^2\varphi}{r^2\sin\theta}\frac{\partial}{\partial\theta} - \frac{\cos^2\theta\sin\varphi\cos\varphi}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi} - \frac{\sin\varphi\cos\varphi}{r^2}\frac{\partial}{\partial\varphi},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial z^2} &= \left(\frac{\partial}{\partial z}\right)^2 \\
&= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\cdot\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) \\
&= \left(\cos\theta\frac{\partial}{\partial r}\right)\cdot\left(\cos\theta\frac{\partial}{\partial r}\right) + \left(-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\cdot\left(-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) \\
&+ \left(\cos\theta\frac{\partial}{\partial r}\right)\cdot\left(-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) + \left(-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\cdot\left(\cos\theta\frac{\partial}{\partial r}\right) \\
&= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta},
\end{aligned}
$$

が得られるから，全て足すと，

$$
\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} = 0 \tag{26}
$$

となることが分かった．